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Vacuum geometry

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Abstract: We analyse symmetry breaking in general gauge theories paying particular attention to the underlying geometry of the theory. In this context we find two natural metrics upon the vacuum manifold: a euclidean metric associated with the scalar sector, and another generally inequivalent metric associated with the gauge sector. Physically, the interplay between these metrics gives rise to many of the non-perturbative features of symmetry breaking.

Keywords: Solitons Monopoles and Instantons, Spontaneous Symmetry Breaking, Gauge Symmetry.

Contents

1. Introduction

Non-abelian gauge theories are the modern setting for theories of particle interaction. Both the strong and electroweak interactions are described by such theories. For electroweak interactions the full gauge symmetry is not apparant at low energies, with it assumed to be broken in some high energy phase transition. This same concept of symmetry breaking may be taken further to unify the strong and electroweak interactions.

Central to the symmetry breaking scheme is the concept of a vacuum. The symmetry is broken by the coupling between this vacuum and the original gauge fields. This coupling induces mass for the gauge fields associated with the broken symmetries, whilst the gauge fields associated with the residual symmetries do not couple, remain massless and form the residual gauge theory.

However, gauge symmetry breaking has further implications. Their structure implies the existence of non-perturbative effects, which imply specific physical consequences for the theory under consideration. Examples of non-perturbative effects include sphaleron processes, topological vortices, topological monopoles and dynamically stable vortices. Their spectrum and properties are very model dependant, depending upon the particular pattern of symmetry breaking.

In this paper we are concerned with the specific geometric structure of the vacuum manifold and how it relates to the spectrum and properties of many non-perturbative effects. Our approach is to associate two natural homogenous metrics on the vacuum manifold. One metric associated with the structure of the scalar sector, the other associated with the gauge sector. Comparison of the relative geodesic structures determines many of the non-perturbative features of the theory.

It transpires that the boundary conditions for many non-perturbative solutions depends upon the comparative geodesic structure of the scalar and gauge metrics. Embedded vortices correspond to mutually geodesic circles, whilst embedded monopoles correspond to mutually geodesic two-spheres and sphalerons correspond to mutually geodesic three spheres. Furthermore, this approach also relates the Aharanov-Bohm scattering of embedded vortices to the holonomy of geodesics with respect to the gauge sector metric.

One should note that the material in this paper stems from a similar geometrical analysis of the Weinberg-Salam model [1]. For illustration we perform the analysis here in parallel to that.

The plan of this paper is as follows. We firstly review gauge symmetry breaking using a notation compatible with the latter sections of this paper. We then discuss the associated gauge and scalar metrics. Finally we apply this discussion to the determination of non-perturbative solutions and their properties in general gauge theories.

2. Gauge symmetry breaking

Consider when a gauge symmetry G is broken to a residual symmetry H. A lagrangean $\mathcal{L}$ describes the interaction of the gauge fields A^μ with scalar fields Φ. These gauge fields A^μ take values in the Lie algebra $\mathcal{G}$, whilst the scalar fields take values in the vector space V. Minimisation of this lagrangean yields a vacuum with gauge invariance under only the subgroup H.

The gauge symmetry G has an action upon the scalar fields, described by the representation D. Correspondingly the Lie algebra also has a derived action upon the scalar fields, described by the derived representation d such that $D(e^X) = e^{d(X)}$ for all $X \in \mathcal{G}$.

The interaction of the scalar-gauge sector is then represented by the lagrangean

$$\mathcal{L} = -\frac{1}{4}\langle F^{\mu\nu}, F_{\mu\nu}\rangle + \mathcal{D}^\mu\Phi^\dagger\,\mathcal{D}_\mu\Phi - V[\Phi]\,, \tag{2.1}$$

with field tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + [A^\mu, A^\nu]\,, \tag{2.2}$$

covariant derivative

$$\mathcal{D}^\mu = \partial^\mu + d(A^\mu)\,, \tag{2.3}$$

and $\langle\cdot,\cdot\rangle$ the inner product on $\mathcal{G}$.

There is some freedom in the choice of non-degenerate inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{G}$, constrained to be invariant under the adjoint action $\mathrm{Ad}(G)$. We parameterise the possible inner products by splitting the Lie algebra $\mathcal{G}$ into its commuting subalgebras $\mathcal{G} = \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_n$ and then defining the inner product $\langle\cdot,\cdot\rangle$ to consist of a linear combination of the inner products on each $\mathcal{G}_l$. The parameters describing the linear combination of inner products then specify the relevant length scales between the commuting components of $\mathcal{G}$. We shall show that these are related to the gauge coupling constants.

Explicitly the general $\mathrm{Ad}(G)$-invariant inner product on $\mathcal{G}$ is

$$\left\langle \sum_l X_l, \sum_l Y_l \right\rangle = \sum_l s_l \{X_l, Y_l\}_l\,, \qquad X_l, Y_l \in \mathcal{G}_l\,, \tag{2.4}$$

with real, positive parameters $\{s_1, \cdots, s_n\}$ and

$$\{X_l, Y_l\}_l = -p\ \mathrm{Re}[\mathrm{tr}(X_l Y_l)]\,, \qquad \mathcal{G}_l \text{ simple}\,, \tag{2.5}$$

$$\{X_l, X_l\}_l = 1\,, \qquad \exp(2\pi X_l) = 1\,, \qquad \mathcal{G}_l \cong \mathrm{u}(1)\,. \tag{2.6}$$

Taking an orthonormal basis $\{X_l^1, \ldots, X_l^{\dim(\mathcal{G}_l)}\}$ for $\mathcal{G}_l$ with respect to $\{\cdot,\cdot\}_l$, the unit norm generators with respect to $\langle\cdot,\cdot\rangle$ are

$$\| q_l(s_1, \cdots s_n) X_l^i \| = 1 \qquad \text{(no sum)}. \tag{2.7}$$

With these generators the covariant derivative explicitly takes the form

$$\mathcal{D}^\mu = \partial^\mu + \sum_l q_l A_l^{i\mu} X_l^i\,, \tag{2.8}$$

from which the scales $\{q_1, \cdots, q_n\}$ are interpreted as the corresponding gauge coupling constants.

Symmetry breaking is seen through minimisation of the lagrangean (2.1), one solution of which is the vacuum

$$\Phi(x) = \Phi_0\,, \qquad A^\mu = 0\,, \tag{2.9}$$

for some arbitrary Φ_0 minimising $V[\Phi]$. The other minima of eq. (2.1) collectively give rise to the vacuum manifold of degenerate equivalent solutions

$$M = D(G)\Phi_0\,, \tag{2.10}$$

a homogenous manifold contained in V.

Around the vacuum in eq. (2.9) the residual field theory has gauge symmetry H. The effect of the interaction between this vacuum and the gauge fields is to give mass to those gauge fields not associated with the generators of H. However the basis $\{X_l^i\}$ does not generally correspond to a basis associated with mass eigenstates. To find this basis of mass eigenstates one must consider an orthogonal transformation of the basis vectors $\{X_l^i\}$.

We shall use a gauge equivalence argument to find this basis of mass eigenstates. Firstly we split the gauge bosons into massless and massive families associated with the decomposition

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{M} . \tag{2.11}$$

This has the algebraic property

$$\mathrm{Ad}(H)\mathcal{H} \subseteq \mathcal{H} , \qquad \mathrm{Ad}(H)\mathcal{M} \subseteq \mathcal{M} , \tag{2.12}$$

which physically represents that the massive gauge bosons are not equivalent to the massless gauge bosons under H.

Massless gauge bosons are generated by elements of $\mathcal{H}$. These split into gauge eigenstate families corresponding to the decomposition

$$\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m , \tag{2.13}$$

of $\mathcal{H}$ into commuting subalgebras. The space of zero mass eigenstates is then

$$\sum_j e_j Y_j^i B_j^{i\mu} , \tag{2.14}$$

with $\{e_j Y_j^1, \ldots, e_j Y_j^{\dim(\mathcal{H}_j)}\}$ forming an orthonormal basis for each $\mathcal{H}_j$ and e_j the corresponding scale. The massless gauge fields are denoted by $B_j^{i\mu}$.

Massive gauge bosons are generated by elements of $\mathcal{M}$. This splits into mass-degenerate families corresponding to the decomposition of

$$\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_l , \tag{2.15}$$

into $\mathrm{Ad}(H)_G$ irreducible subspaces of $\mathcal{M}$. Physically the H-equivalent gauge bosons are degenerate in mass, which split the spectrum of H-equivalent gauge boson generators into mass equivalent families. The space of massive eigenstates is then

$$\sum_f g_f Z_f^i W_f^{i\mu} , \tag{2.16}$$

with $\{g_f Z_f^1, \ldots, g_f Z_f^{\dim(\mathcal{M}_f)}\}$ forming an orthonormal basis for $\mathcal{M}_f$, with scale g_f and $W_f^{i\mu}$ the corresponding massive gauge fields.

Taken together eqs. (2.14) and (2.16) specify the covariant derivative to be

$$\mathcal{D}^\mu = \partial^\mu + \sum_j e_j Y_j^i B_j^{i\mu} + \sum_f g_f Z_f^i W_f^{i\mu} , \tag{2.17}$$

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explicitly composed of the residual and broken parts. Again the scales $\{e_j\}$ and $\{g_f\}$ are interpreted as the associated gauge coupling constants. They depend upon the scales $\{q_i\}$, with the explicit dependance determined by the orthogonal transformation that relates the $\{X_i\}$ basis of $\mathcal{G}$ to the mass eigenstate basis above.

3. Vacuum geometry

For both the scalar and gauge sectors we now explicitly calculate their associated metrics. We also calculate the associated isometry and isotropy groups for each, and from these groups specify the corresponding geodesic structures.

Having obtained two inequivalent homogenous metrics on the vacuum manifold we then compare their structure. We relate their isometry and isotropy groups, and determine when curves are mutually geodesic with respect to both metrics.

3.1 Gauge sector

The main structure associated with the gauge sector is the inner product $\langle \cdot, \cdot \rangle$ of eq. (2.4). It specifies several important related features of the gauge sector. First of all it defines the gauge kinetic term in eq. (2.1), introducing the gauge coupling constants as the relative scales. Secondly it stipulates the massive and massless gauge boson generators to be the orthogonal subspaces $\mathcal{M}$ and $\mathcal{H}$ of eq. (2.11). Finally it renders the degenerate mass eigenstate families $\mathcal{M}_i$ to be mutually orthogonal.

We shall use this inner-product to define the gauge-sector metric. The definition is achieved by associating the massive generators $\mathcal{M}$ with tangent spaces to the vacuum manifold in a natural way. Then the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{M}$ induces a homogenous metric on the vacuum manifold. We find the corresponding isometry group of this metric to be the gauge group G and the isotropy group to be the residual symmetry H.

Explicitly, observe that the tangent space to M at Φ_0 may be expressed

$$T_{\Phi_0} M = d(\mathcal{M})\Phi_0 \,. \tag{3.1}$$

More generally the corresponding tangent space at $\Phi = g\Phi_0 \in M$ is, for any $g \in G$,

$$T_{\Phi} M = D(g) T_{\Phi_0} M = d(\mathrm{Ad}(g)\mathcal{M})\Phi \,, \tag{3.2}$$

Transitivity over M guarantees a natural isomorphism between any tangent space and some $\mathrm{Ad}(g)\mathcal{M}$.

Using the isomorphism implied by eq. (3.2), the inner product $\langle \cdot, \cdot \rangle$ associates a corresponding metric on M

$$h(X_1\Phi, X_2\Phi)_{\Phi} = \langle X_1, X_2 \rangle \,, \qquad X_1, X_2 \in \mathrm{Ad}(g)\mathcal{M} \,. \tag{3.3}$$

The precise form is parameterised by the scales $\{g_f\}$ of eq. (2.16).

This metric has the gauge symmetry group G of isometries

$$h(D(g)T_1, D(g)T_2)_{D(g)\Phi} = h(T_1, T_2)_\Phi\,, \qquad g \in G\,. \tag{3.4}$$

More precisely, by eq. (3.2) the action of $g \in G$ upon $h(\cdot,\cdot)$ is

$$h(D(g)T_1, D(g)T_2)_{D(g)\Phi} = \langle \mathrm{Ad}(g)X_1, \mathrm{Ad}(g)X_2\rangle = \langle X_1, X_2\rangle = h(T_1, T_2)_\Phi\,. \tag{3.5}$$

The above isometries represent the maximal group acting upon $\mathcal{G}$ that leaves $\langle\cdot,\cdot\rangle$ invariant.

The isotropy group of this isometry group at the point Φ_0 in M is the subgroup H such that

$$D(H)\Phi_0 = \Phi_0\,, \tag{3.6}$$

which gives the isomorphism

$$M \cong \frac{G}{H}\,. \tag{3.7}$$

Thus we recover the familiar relation for the vacuum manifold, but now explicitly associated with the gauge sector metrical structure.

Given the above isotropy and isometry properties of the gauge metric we can use the isomorphism (3.7) to calculate the corresponding geodesics upon M. This geodesic structure follows from some results of differential geometry. Specifically, these results examine the geodesic structure on the coset space, but may be simply carried back to the vacuum manifold to give the results that we require. The relevant results are quoted here, although the full approach is described in detail by Kobayashi and Nomizu [2, chapter X].

The gauge sector geodesic structure is simply

the geodesics on M with respect to the metric $h(\cdot,\cdot)$, passing through Φ_0 are:

$$\gamma_X = \{\exp(Xt)\Phi_0 : t \in \mathbb{R}\}\,, \tag{3.8}$$

with $X \in \mathcal{M}$.

3.2 Scalar sector

The structure associated with the scalar sector is a vector space of scalar field values V equipped with a real euclidean inner product $\mathrm{Re}[\Psi^\dagger\Phi]$. Regarding the vacuum manifold as embedded within the vector space of scalar field values, a natural metric may be induced on M by specifying its form on each tangent space to be that of the euclidean inner product.

Explicitly, regard the tangent space to M at $\Phi \in M$, specified by eq. (3.2), as a vector subspace of V

$$T_\Phi M \subset V\,. \tag{3.9}$$

Then a corresponding metric is induced from the real euclidean inner product

$$g(T_1, T_2)_\Phi = \mathrm{Re}[T_1^\dagger T_2]\,, \qquad T_1, T_2 \in T_\Phi M\,. \tag{3.10}$$

This metric has a group I of isometries acting on V by the Δ-representation

$$g(\Delta(a)T_1, \Delta(a)T_2)_{\Delta(a)\Phi} = g(T_1, T_2)_\Phi\,, \qquad a \in I\,. \tag{3.11}$$

consisting of those elements of $\mathrm{GL}(V)$ that leave the metric invariant. Since $I \subset \mathrm{GL}(V)$ the representation of I upon V is induced by

$$\Delta(I) \subset f(\mathrm{GL}(V))\,, \tag{3.12}$$

with f the fundamental representation of $\mathrm{GL}(V)$ upon V.

The isotropy group of I upon M at the point Φ_0 is the subgroup J such that

$$\Delta(J)\Phi_0 = \Phi_0\,, \tag{3.13}$$

This gives the isomorphism

$$M \cong \frac{I}{J}\,, \tag{3.14}$$

representing a second isomorphism with a coset space, and separate from the usual one associated with the gauge sector in eq. (3.7).

The scalar coset space isomorphism found above in eq. (3.14) is explicitly related to the scalar sector structure. In general the gauge group G is a subgroup of the isotropy group I, with the representations coinciding so that $D(G) \subseteq \Delta(I)$. Symmetries not contained in G are interpreted as global symmetries of the vacuum manifold not apparent in the full gauge theory.

We now prove $D(G) \subseteq \Delta(I)$. This follows from the gauge invariance of the lagrangean (2.1), so that any $g \in G$ preserves the euclidean norm,

$$\mathrm{Re}[\{D(g)T\}^\dagger D(g)T\}] = \mathrm{Re}[T^\dagger T]\,, \tag{3.15}$$

for all T in any $T_\Phi M$. Thus $D(g)^\dagger D(g) = 1$, implying

$$g(D(g)T_1, D(g)T_2)_{a\Phi} = \mathrm{Re}[\{D(g)T_1\}^\dagger D(g)T_2] \tag{3.16}$$
$$= g(T_1, T_2)_\Phi. \tag{3.17}$$

Proving our statement.

As for the gauge sector the importance of isomorphism (3.14) is the calculational use of the isotropy and isometry properties to determine the associated geodesic structure upon M. Since our method is the same for both the scalar and gauge sectors this allows direct comparison to be drawn between them.

Associated with the isomorphism (3.14) is the decomposition

$$\mathcal{I} = \mathcal{J} \oplus \mathcal{N}\,, \tag{3.18}$$

where $\mathcal{N}$ is associated with the tangent space to M at Φ_0

$$T_{\Phi_0}M = \delta(\mathcal{N})\Phi_0\,, \tag{3.19}$$

with δ the derived representation of Δ. The relevant inner product, which defines the decomposition (3.18), is defined by:

> *associated with the I-invariant metric $g(\cdot,\cdot)$ upon M is an inner product upon $\mathcal{N}$*
>
> $$\{X,Y\} = g_{\Phi_0}(X\Phi_0, Y\Phi_0) = \mathrm{Re}[\{X\Phi_0\}^\dagger Y\Phi_0]\,; \tag{3.20}$$
>
> *extendable to a unique inner product on $\mathcal{I}$ such that $\{\mathcal{J},\mathcal{N}\}=0$.*

Geodesic structure then follows analogously to eq. (3.8):

> *the geodesics on M with respect to the scalar metric $g(\cdot,\cdot)$, passing through Φ_0 are:*
>
> $$\gamma_X = \{\exp(Xt)\Phi_0 : t\in\mathbb{R}\}\,, \tag{3.21}$$
>
> *for elements $X\in\mathcal{N}$.*

One should note that this geodesic structure does not in general coincide with that of the gauge sector, although it may do for certain scales $\{q_i\}$. The geodesic structure on M with respect to G/H may be interpreted as homogenously squashed with respect to that from I/J. More exactly we have shown that G is a subgroup of the isometry group I, hence the associated gauge metric has correspondingly less invariance.

For instance S^3 admits a family of $\mathrm{SU}(2)\times\mathrm{U}(1)$ invariant metrics that continuously deforms to an $\mathrm{SU}(2)\times\mathrm{SU}(2)$ invariant metric; and S^7 admits a family of $\mathrm{SO}(5)\times\mathrm{SU}(2)$ invariant metrics that continuously deform to the $\mathrm{SO}(8)$ invariant metric.

3.3 Scalar-gauge geometry

In summary, there are two inequivalent metrics on the electroweak vacuum manifold associated with the scalar and gauge sectors. We shall now determine how the structure of these metrics relate to each other. Comparing the respective symmetry groups determines those symmetries that are shared. These shared symmetries define submanifolds whose geodesics are mutually geodesic with respect to both metrics.

The scalar and gauge metrics, $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$, have the following isometry group decompositions with respect to their isotropy groups

$$\mathcal{I} = \mathcal{J}\oplus\mathcal{N}\,,\qquad \mathcal{G} = \mathcal{H}\oplus\mathcal{M}\,, \tag{3.22}$$

where their group structures are related by

$$H\subset J\,,\qquad G\subset I\,,\qquad \mathcal{M}\subset\mathcal{I}\,, \tag{3.23}$$

and the representations of G and I are coincident on the intersection

$$D(G)\subseteq\Delta(I)\,. \tag{3.24}$$

Also the tangent space to M at Φ_0 is related to $\mathcal{M}$ and $\mathcal{N}$ by

$$T_{\Phi_0}M = d(\mathcal{M})\Phi_0 = \delta(\mathcal{N})\Phi_0 \,. \tag{3.25}$$

It is important to understand how the metrics $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$ are related. It transpires that this is through the decomposition of $\mathcal{M}$ into its $\mathrm{Ad}(H)_G$ irreducible subspaces

$$\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_k \,, \tag{3.26}$$

which defines a decomposition of the tangent space into its $D(H)$ irreducible subspaces $T^i_{\Phi_0}M = d(\mathcal{M}_i)\Phi_0$

$$T_{\Phi_0}M = T^1_{\Phi_0}M \oplus \cdots \oplus T^k_{\Phi_0}M \,. \tag{3.27}$$

Then at $\Phi_0 \in M$ the metrics $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$ are related by a bilinear transformation of $T_{\Phi_0}M$ taking the particular form

$$g(\sum_i X_i\Phi_0, \sum_j Y_j\Phi_0) = \sum_{ij} \lambda_i\lambda_j h(X_i\Phi_0, Y_j\Phi_0) \,, \tag{3.28}$$

where

$$\lambda_i = \frac{g(X_i\Phi_0, X_i\Phi_0)}{h(X_i\Phi_0, X_i\Phi_0)} \tag{3.29}$$

and $X_i\Phi_0 \in T^i_{\Phi_0}M$.

Result (3.28) is established by noting that since both metrics are bilinear they are related by an element $A \in GL(T_{\Phi_0}M)$

$$g(T_1, T_2)_{\Phi_0} = h(AT_1, AT_2)_{\Phi_0} \,, \qquad A \in GL(T_{\Phi_0}M). \tag{3.30}$$

Now observe $D(H) \subseteq GL(T_{\Phi_0}M)$, is the *maximal* subgroup of $GL(T_{\Phi_0}M)$ under which both metrics are invariant. Then $D(h)A = AD(h)$ for all $h \in H$. It follows that the eigenspaces of A are the irreducible spaces of $T_{\Phi_0}M$ under $D(H)$, from which one associates eqs. (3.26) and (3.27) and establishes the result (3.28) and (3.29). This may be easily generalised to all $\Phi \in M$ by considering the action of G on eq. (3.28).

The decomposition (3.27) also describes the geodesic structure of M with respect to $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$ in a rather nice way. Applying eqs. (3.21) and (3.8), the submanifolds

$$M^i = D(\exp(\mathcal{M}_i))\Phi_0 \,, \qquad T_{\Phi_0}M^i = T^i_{\Phi_0}M \tag{3.31}$$

are the *maximal* totally geodesic submanifolds of M with respect to *both* metrics $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$. They naturally decompose M into totally geodesic component parts.

One should note that the $\mathrm{Ad}(H)_G$-irreducible subspaces of $\mathcal{M}$ also relate to the mass eigenstates of the massive gauge bosons. It is interesting how the same decomposition arises in two, at first sight, apparently unrelated areas of the gauge theory; however the association is that both are governed by the action of H within the broken theory.

4. Physical implications

In summary of the previous section: there are two homogenous metrics on the vacuum manifold associated with the scalar and gauge sectors of the scalar-gauge theory. These two sectors induce metrics upon the vacuum manifold associated with the symmetries of their respective sector. Generally the geodesics with respect to these two metrics coincide on totally geodesic submanifolds. These totally geodesic submanifolds are embedded in M such that at any point their collection of tangent spaces do not intersect and collectively from the tangent space to M at that point.

Given this structure we examine its relation to the spectrum of non-perturbative solitonic-type solutions present for a general symmetry breaking. It transpires that the embedded vortices correspond to the mutually geodesic paths, whilst monopoles correspond to mutually geodesic two-spheres and sphalerons to mutually geodesics three-spheres. This approach also interprets the scattering of embedded vortices in terms of the holonomy of their respective geodesics. Furthermore the dynamical stability of a semi-local vortices, in their respective semi-local limit, is seen to correspond to an extreme limit of the gauge metric.

4.1 Embedded vortices

Embedded vortices correspond to Nielsen-Olesen vortices embedded in a general symmetry breaking [3]. As such their boundary conditions define circular paths on the vacuum manifold. Thus it might be expected that their spectrum and properties should correspond to the geometry of the vacuum manifold. This is what we find. Their boundary conditions correspond to the paths that are mutually geodesic with respect to *both* metrics.

Formally, an embedded vortex is defined by the embedding

$$\begin{array}{ccc} G & \to & H \\ \cup & & \cup \\ \mathrm{U}(1) & \to & 1, \end{array} \tag{4.1}$$

with the general ansatz

$$\Phi(r,\theta) = f_{\mathrm{NO}}(r)e^{X\theta}\Phi_0\,, \tag{4.2}$$

$$\underline{A}(r,\theta) = \frac{g_{\mathrm{NO}}(r)}{r}X\underline{\hat{\theta}}\,, \tag{4.3}$$

where $X \in \mathcal{G}$ is the vortex generator. One may consider only $X \in \mathcal{M}$, since these minimise the magnetic energy [4]. Thus one considers only Ansätze with boundary conditions geodesic with respect to the gauge metric $h(\cdot,\cdot)$.

The above vortex ansatz is a solution provided that [5]

(i) The scalar field must be single-valued. Hence the boundary conditions describe a closed geodesic with $e^{2\pi X}\Phi_0 = \Phi_0$.

(ii) The ansatz is a solution to the equations of motion; then fields in the vortex do not induce currents perpendicular (in Lie algebra space) to it [3]. This may be equivalently phrased as [4]: X is a vortex generator if $\mathrm{Re}[(X\Phi_0)^\dagger X^\perp \Phi_0] = 0$ for all $X^\perp$ such that $\langle X^\perp, X\rangle = 0$.

Condition (ii) can be conveniently restated in terms of the corresponding metrics:

X is a vortex generator if the associated tangent vector $T = X\Phi_0$ satisfies $g(T, T^\perp) = 0$ for all $T^\perp$ such that $h(T, T^\perp) = 0$.

Referring to the discussion around eq. (3.27), we see that T must lie in one of the $D(H)$-irreducible subspaces of $T_{\Phi_0}M$ that relate the two metrics. Namely X is an element of any of the $\mathrm{Ad}(H)_G$-irreducible subspaces of $\mathcal{M}$ in the decomposition

$$\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n\,. \tag{4.4}$$

It is interesting that the geodesics defined from the $\mathrm{Ad}(H)$-irreducible subspaces $\mathcal{M}_i$ are the only geodesics which are *simultaneously geodesic with respect to both metrics*. From a mathematical point of view this is because these geodesics define submanifolds of the vacuum manifold with coincident scalar and gauge metrics (to an overall factor). From a physical point of view, this may be interpreted as a minimisation of both the scalar and gauge sectors of the action integral.

Vortices within the same family $\mathcal{M}_i$ have the same stability properties because of their gauge equivalence. For topologically stable embedded vortices the associated geodesic $\gamma_X(t) = D(\exp(Xt)\Phi_0$ has a conserved topological charge corresponding to the element of $\pi_1(G/H)$ it represents. Dynamically stable vortices are discussed next.

4.2 Dynamical stability

Embedded vortices may become dynamically stable if there exists a limit of the coupling constants $\{q_i\} \to \{\tilde{q}_i\}$ where the symmetry breaking takes the form [7]

$$G_g \times \mathrm{U}(1)_l \to H\,, \qquad \text{with } H \cap \mathrm{U}(1)_l = \mathbf{1}\,. \tag{4.5}$$

In this limit there exists for small scalar potentials dynamically stable semi-local vortices [6], and by continuity their stability persists for nearby values of the coupling constants. Here we show that this is related to an extreme limit of the gauge sector metric. In this limit it becomes only well defined on a circle corresponding to the boundary conditions of the semi-local vortex.

Algebraically eq. (4.5) is related to the decomposition of $\mathcal{M}$ into its $\mathrm{Ad}(H)_G$ irreducible subspaces [4]:

> *when $\mathcal{M}_l$ represents a non-trivial projection of $\mathcal{C}$, the centre of $\mathcal{G}$, onto $\mathcal{M}$ then that family may define dynamically stable vortices.*

Such $\mathcal{M}_l$'s are one-dimensional, with their vortices invariant under H.

This qualitative difference of the gauge theory when the coupling constants take values $\{\tilde{q}_i\}$ is also apparent in the form of the gauge sector metric $h(\cdot,\cdot)$. When the coupling constants take the value $\{\tilde{q}_i\}$, the inner product $\langle\cdot,\cdot\rangle$ becomes ill defined; being well defined only on the subalgebra $\mathrm{u}(1)_l$, where it takes the form

$$\langle X, Y\rangle_{q_l} = -\frac{1}{q_l^2}\,\mathrm{tr}\,X\,\mathrm{tr}\,Y\,. \tag{4.6}$$

Referring to the discussion around eqs. (3.3) and (3.31), we see that the metric is only defined upon the one-dimensional submanifold $M_l = D(\mathrm{U}(1)_l)\Phi_0 \subset M$, where it takes the form

$$h_{q_l}(T_1, T_2)_\Phi = \langle X_1, X_2\rangle_{q_l}\,, \qquad T_i = d(X_i)\Phi \in T_\Phi M_l\,, \tag{4.7}$$

with $a \in \mathrm{U}(1)_l$, relating $\Phi = D(a)\Phi_0$.

The metric $h(\cdot,\cdot)$ thus picks out a prefered submanifold $M_l \subset M$ over which it is well defined. Physically, this submanifold corresponds to those points that may be reached by a gauge transformation from Φ_0. Other points within the vacuum manifold may only be reached by a global transformation. It is precisely such submanifolds that define the boundary conditions for vortices that may be dynamically stable.

4.3 Combination electroweak vortices

Recall that the two metrics are related by

$$g\left(\sum_i X_i\Phi_0, \sum_j Y_j\Phi_0\right) = \sum_{ij}\lambda_i\lambda_j h(X_i\Phi_0, Y_j\Phi_0)\,, \tag{4.8}$$

where

$$\lambda_i = \frac{g(X_i\Phi_0, X_i\Phi_0)}{h(X_i\Phi_0, X_i\Phi_0)}\,, \tag{4.9}$$

which is the reason for claiming the coincidence of the metrics on certain submanifolds defined by the $\mathrm{Ad}(H)_G$ irreducible subspaces $\mathcal{M}_i$. In sec. (3.3) we then showed that these $\mathcal{M}_i$'s contain the vortex generators, since to define solutions the scalar boundary conditions must be mutually geodesic with respect to both metrics.

The constants $\{\lambda_i\}$, which define the relative scales of the two metrics, are dependant upon the gauge coupling constants $\{q_i\}$. Should coupling constants take critical values such that $\lambda_i = \lambda_j$, the two metrics $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$ become coincident

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(to a factor) on a larger submanifold $M_{ij} = D(\exp(\mathcal{M}_i \oplus \mathcal{M}_j))\Phi_0$, containing both M_i and M_j. Then one concludes that the vortex generators are members of a larger family $\mathcal{M}_i \oplus \mathcal{M}_j$, and one refers to these extra vortices as *combination* vortices — interpreting them as two vortices in the original families combined together.

For critical coupling constants such that $\lambda_i = \lambda_j$ one of two situations may occur, depending upon the details of the geometry:

(i) The isometry group of the gauge sector metric $h(\cdot,\cdot)$ enlarges. Then vortices in the two families $\mathcal{M}_i$ and $\mathcal{M}_j$ becomes equivalent under global transformations of these newly acquired symmetries.

(ii) The isometry group of the gauge sector metric $h(\cdot,\cdot)$ stays the same. The equivalence of vortices remains the same as the non-critical case; however, the solution set of embedded vortices increases to include the combination vortices.

These two situations relate to the symmetry properties of scattering two vortices. For two different vortices heading towards each other the symmetry of the scattering process causes them to travel along the line joining the centres both before and after they meet. However, when the vortices are the same, as happens in (i) above, they may scatter at right angles to the original path, with the natural motion passing through a toroidal charge two configuration. This charge two configuration corresponds to a combination vortex.

Although the combination vortex solutions are not solutions for non-critical coupling constants, it may be possible that static deformed solutions persist. Such solutions would consist of perturbed solutions around the embedded vortex, with the perturbations determined by substitution into the field equations.

4.4 Non-abelian Aharanov-Bohm scattering

Associated with the magnetic flux of a vortex is the Wilson line integral

$$\mathrm{U}(\theta) = P\exp\left(\int_0^\theta \mathbf{A}\cdot d\mathbf{l}\right) \subset G\,, \tag{4.10}$$

at infinite radius. Then $\mathrm{U}(\theta)$ dictates the parallel transport of matter fields around a vortex, with fermionic fields Ψ transported to $\mathrm{U}(2\pi)\cdot\Psi$ around a vortex. By diagonalisation one then associates components Ψ_i with phase shifts ξ_i.

The Wilson line integral is fundamental to understanding the interaction of a vortex with charged matter fields [8]. Non-trivial fermionic components Ψ_i interact with the vortex by an Aharanov-Bohm cross section

$$\frac{d\sigma}{d\theta} = \frac{1}{2\pi k}\frac{\sin^2(\xi_i/2)}{\sin^2(\theta/2)}\,, \tag{4.11}$$

whilst trivial components $e^{2\pi\xi_k} = 1$ interact by an Everett cross section [9].

Substitution of eq. (4.3) into eq. (4.10) gives the Wilson line integral for an embedded vortex, with generator $X \in \mathcal{M}$,

$$\mathrm{U}(\theta) = \exp(X\theta) \subset G\,. \tag{4.12}$$

Vortex boundary conditions restrict $\mathrm{U}(2\pi) \in H$. Also, by eq. (4.2) the asymptotic scalar field satisfies $\Phi_\infty(\theta) = D(\mathrm{U}(\theta))\Phi_0$. For a given fermion representation this specifies completely the scattering of a fermion component off an embedded vortex.

In [8] the Wilson line integral is related to holonomy, referring to non-trivial parallel transport around the vortex. In fact, we now show that this holonomy refers to precisely the *parallel transport with respect to the gauge sector metric around its closed geodesics*.

Parallel transport with respect to the gauge sector metric $h(\cdot,\cdot)$ on M is related to the symmetry groups G and H by [2]:

> *the parallel transport of tangent vector $T \in T_{\Phi_0}M$ along the geodesic $\gamma_X(t) = D(\exp(Xt))\Phi_0$ to the point $\gamma_X(s)$ is*
>
> $$T' = D(\exp(Xs))T\,. \tag{4.13}$$

The holonomy is non-trivial when $\gamma_X(s) = \gamma_X(0)$ and $T' \neq T$.

Hence geodesic parallel transport and the Wilson line integral are related. For a tangent vector $T \in T_{\Phi_0}M$, its parallel transport to $\gamma_X(\theta)$ is

$$T' = D(\mathrm{U}(\theta))T\,. \tag{4.14}$$

Interpreting tangent vectors as components of the scalar field we see that eq. (4.14) also refers to a global transformation of G upon the scalar field. The same global transformation is relevant to the parallel transport of a fermion representation along the geodesic. Thus we see that the Wilson line integral refers to parallel transport along a geodesic with respect to the gauge sector metric $h(\cdot,\cdot)$. One should recall also that the geodesics relevant to vortices are geodesic with respect to both metrics, however it is only the gauge sector holonomy that is relevant to scattering.

4.5 Embedded monopoles

Embedded, or fundamental, monopole solutions are t'Hooft-Polyakov monopoles embedded within the gauge theory. To avoid ambiguity between embedded SU(2) or embedded SO(3) monopoles, the embedding is described in terms of the algebras

$$\begin{array}{ccc} \mathcal{G} & \to & \mathcal{H} \\ \cup & & \cup \\ \mathrm{su}(2) & \to & \mathrm{u}(1)\,, \end{array} \tag{4.15}$$

with the general ansatz

$$\underline{\Phi}(\underline{r}) = f_{\text{mon}}(r)\underline{\hat{r}}, \qquad A_a^\mu(\underline{r}) = \frac{g_{\text{mon}}(r)}{r}\epsilon_{ab}^\mu X_b\,, \tag{4.16}$$

where $f_{\text{mon}}(r)$ and $g_{\text{mon}}(r)$ are the usual monopole profile functions. Notationally, we treat Φ a vector within $\mathbb{R}\cdot D(\text{SU}(2))\Phi_0$ and use an orthonormal basis $\{X_1, X_2, X_3\}$ for su(2), with $X_3 \in \mathcal{H}$. One may consider only $X_1, X_2 \in \mathcal{M}$ since these minimise the magnetic energy. Thus one considers only Ansätze with boundary conditions geodesic two-spheres with respect to the $h(\cdot,\cdot)$ metric.

By relating the above embedding to the spectrum of embedded vortices, one arrives at the following result [4]:

embedded monopoles are defined by two generators X_1, X_2 such that

$$X_1, X_2 \in \mathcal{M}_i\,, \qquad \text{with } [X_1, X_2] \in \mathcal{H}\,. \tag{4.17}$$

Hence the gauge equivalence classes of embedded monopoles are similar to those of embedded vortices, but with extra constraints.

Boundary conditions for the embedded monopole define a two-dimensional spherical submanifold $D(\text{SU}(2))\Phi_0 \subset M$. Analogous to embedded vortices these submanifolds are *simultaneously geodesic with respect to both metrics*. However, in contrast to vortices, there may be other geodesic two-spheres not associated with embedded monopoles: for instance the Sphaleron configurations, which we discuss next.

4.6 Embedded sphalerons

Analogously to embedded vortices and monopole one may embed sphalerons [10] in a general symmetry breaking. To avoid ambiguity between SU(2) and SO(3) Sphalerons the embedding is discussed in terms of the algebras. For an su(2) embedded Sphalerons the embedding is

$$\begin{array}{ccc} \mathcal{G} & \to & \mathcal{H} \\ \cup & & \cup \\ \text{su}(2) & \to & \mathbf{1}\,; \end{array} \tag{4.18}$$

whilst for an su(2) $\oplus$ u(1) Sphaleron the embedding is

$$\begin{array}{ccc} \mathcal{G} & \to & \mathcal{H} \\ \cup & & \cup \\ \text{su}(2)\oplus\text{u}(1) & \to & \text{u}(1)_{\text{diag}}\,. \end{array} \tag{4.19}$$

The corresponding ansätze are simply the usual ones embedded in a larger theory.

By a similar analysis to that of monopoles one may conclude that an embedded su(2) sphaleron corresponds to simultaneously geodesic three-spheres; whilst an embedded su(2) $\oplus$ u(1) sphaleron corresponds to a geodesic three-sphere submanifold with respect to $g(\cdot,\cdot)$ and a squashed three-sphere submanifold with respect to $h(\cdot,\cdot)$. Contained within this embedded three-sphere is a two sphere that is simultaneously geodesic with respect to both metrics, and has no topology associated with it.

Physically, embedded sphalerons relate to the vacuum structure of the gauge theory. Inequivalent vacua of the quantum field theory are labelled by the elements of $\pi_3(G/H)$ and correspond to the Chern-Simons number of the vacua. Sphalerons represent the midpoint of a sequence of configurations passing from one vacuum to another; they are thus physically relevant when they have an embedding associated with a non-trivial element of $\pi_3(G/H)$.

5. Discussion

In this final section we briefly discuss some extra points of note, and indicate some possible extensions to the work of this paper.

(i) **Simplicity of electroweak theory.** We showed that the maximal symmetry group of any metric upon the vacuum manifold is the isometry group I, with the gauge group $G \subseteq I$. For electroweak theory $I = \mathrm{SU}(2)_I \times \mathrm{SU}(2)_K$ and $G = \mathrm{SU}(2)_I \times \mathrm{U}(1)_Y$. In fact this constitutes the *smallest* dimensional example of such a structure [1].

(ii) **Energetics and curvature.** The metrical structure determines an associated curvature of the vacuum manifold. It seems natural that the energy of embedded vortices should correspond with the sum of the relevant sectional curvatures associated with the scalar and gauge sectors on the particular submanifold of M associated with the scalar boundary conditions of the vortex. Coefficients of this sum would naturally be related to those of the scalar potential, and the value of this sum to the stability of the vortex.

(iii) **Non-abelian zero modes.** The zero modes of a configuration correspond to zero energy deformations of that configuration. Since the relevant configurations correspond to geodesic submanifolds, the zero modes must preserve this property. Thus zero modes should be related to the metrical structure of the vacuum manifold.

(iv) **Mathematics of the scalar metric.** Often the scalar sector metric, associated with I/J, corresponds to a symmetric space. Consequently the scalar sector holonomy is trivial. It would be interesting to determine exactly when a symmetric space is associated, and whether this is always required to give a trivial holonomy to the scalar sector.

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